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General insertion and extension theorems for localic real functions[☆]

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ABSTRACT

In this paper we investigate localic real functions on frames. We provide a necessary and sufficient condition for the insertion of a continuous localic real function between two arbitrary comparable localic real functions. We also establish necessary and sufficient conditions for extending a bounded localic real function from a complemented sublocale to the whole frame.

1. Introduction

Via definition, all localic maps are *continuous* (this has recently been repeated by Vickers [27, p. 23]). However, in point-set topology one sometimes considers maps which are not continuous. In particular, this is the case of real valued functions on a topological space. After having introduced a localic analogue of the concept of an *arbitrary* (not necessarily continuous) real valued function (by Gutiérrez García et al. [12]), we now have new areas of topology to be explored in the world of locales (cf. the very recent paper [8]). The motivation of this paper is the lack of localic variants of two topological results (the first of which involves arbitrary not necessarily continuous real valued function), viz.: *Topological Insertion Theorem* of Blair [5] and Lane [21] and *Topological Extension Theorem* of Mrówka [23]. For the reader's convenience, we shall record those two theorems below. Given a set X, a function $f : X \to \mathbb{R}$ and $t \in \mathbb{R}$, we write: $[f \le t] = \{x \in X : f(x) \le t\}$ and $[f \ge t] = \{x \in X : f(x) \ge t\}$. Two subsets A and B of a topological space X are said to be *completely separated* in X if there is a continuous $f : X \to [0, 1]$ such that $f = 0$ on A and $f = 1$ on B.

Topological Insertion Theorem (*Blair [5] and Lane [21]*)**.** *Let X be a topological space and let $g, h : X \to \mathbb{R}$ be arbitrary (not necessarily continuous) functions. The following are equivalent:*

(1) *There exists a continuous $f : X \to \mathbb{R}$ such that $g \le f \le h$.*
(2) *The sets $[h \le r]$ and $[g \ge s]$ are completely separated in X for all $r < s$ in $\mathbb{R}$.*

Topological Extension Theorem (*Mrówka [23]*)**.** *Let X be a topological space, let S be an arbitrary subspace of X, and let $f : S \to \mathbb{R}$ be a bounded continuous function. The following are equivalent:*

(1) *f has a continuous extension to the whole of X.*
(2) *The sets $[f \le r]$ and $[f \ge s]$ are completely separated in X for all $r < s$ in $\mathbb{R}$.*

For a historical account of these two theorems we refer to Blair and Swardson [6].

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2. Locales, sublocales, localic reals and localic real functions

Useful sources of references for frames are [3,15,25,26]. This section is of a preparatory character. It contains some basic frame-theoretic terminologies with emphasis on sublocales and localic real functions (as recently defined in [12]).

I. Locales. A *frame* or *locale* is a complete lattice L in which

$$a \wedge \bigvee B = \bigvee \{a \wedge b : b \in B\}$$

for all $a \in L$ and $B \subseteq L$. The universal bounds are denoted by 0 and 1 (if necessary, we put 0_L and 1_L). A *frame morphism* is a map $f : L \to M$ between frames which preserves finite meets and arbitrary joins (so it preserves universal bounds). [Historically, the prototype of a frame is the topology $\mathcal{O}X$ of a topological space X. If $f : X \to Y$ is continuous, then $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ determined by $\mathcal{O}f(U) = f^{-1}(U)$ is a frame homomorphism. Thus $\mathcal{O}$ is a contravariant functor from the category Top of topological spaces to the category Frm of frames.]

Each frame morphism $f : L \to M$ has its right adjoint $f_* : M \to L$ satisfying $a \leq f_*(b)$ if and only if $f(a) \leq b$. Hence $f_*(b) = \bigvee \{a \in L : f(a) \leq b\}$ for all $a \in L$ and $b \in M$.

Remark 2.1. If $f : L \to M$ is a frame morphism and $a \in L$, then

$$f(a) = 0 \quad \text{iff} \quad f_*(M) \subseteq \uparrow a.$$

Indeed, if $f(a) = 0 \leq b$, then $f_*(b) \geq a$ for all $b \in M$, i.e. $f_*(M) \subseteq \uparrow a$. And if the latter holds, then $f(a) \leq b$ for all $b \in M$, i.e. $f(a) = 0$.

The map $(\cdot) \wedge a : L \to L$ preserves arbitrary joins and, thus, has a right adjoint $a \to (\cdot) : L \to L$. This means that $x \wedge a \leq b$ iff $x \leq a \to b$. Thus $a \to b = \bigvee \{x \in L : x \wedge a \leq b\}$. The *pseudocomplement* of $a \in L$ is $a^* = a \to 0$. We have $x \leq a^*$ iff $x \wedge a = 0$. In particular, $a \wedge a^* = 0$ and $a \leq a^{**}$. Also, $(\bigvee A)^* = \bigwedge A^*$ (so $(\cdot)^*$ is antitone) where (here and elsewhere)

$$A^* = \{a^* : a \in A\}.$$

II. Sublocales. A subspace Y of a topological space X and the inclusion map $\iota : Y \hookrightarrow X$ define a surjective frame morphism $\mathcal{O}\iota : \mathcal{O}X \to \mathcal{O}Y$ by $\mathcal{O}\iota(U) = U \cap Y$. After [7], surjective frame morphisms were chosen to represent localic analogues of subspaces. Given a surjective frame morphism $h : L \to M$, the frame M is called a *quotient* of L. As observed by Dowker and Papert [7, p. 280], the quotient M can be identified with a subset of L with induced order which is called a *sublocale* of L. [For historical reasons we note that many authors incorrectly consider [15, Exercise II-2.3] as the original source of the characterization of a sublocale as a subset.]

According to [7], a subset $S \subseteq L$ is a *sublocale* of L if it is closed under arbitrary meets (in particular $\bigwedge \varnothing = 1 \in S$) and the map $a \to (\cdot)$ sends S into S for all $a \in L$.

For any $a \in L$, the sets of the form

$$\mathfrak{c}(a) = \uparrow a \quad \text{and} \quad \mathfrak{o}(a) = \{a \to b : b \in L\}$$

are sublocales of L called, respectively, *closed* and *open*.

Remark 2.2 (*cf.* [7,16,24,25])**.** Sublocales can be represented by surjective frame morphisms and vice versa. This one-to-one correspondence is determined as follows: if $f : L \to M$ is onto, then $S_M = f_*(M)$ is a sublocale. On the other hand, a sublocale $S \subseteq L$ determines the surjection $c_S : L \to S$ given by $c_S(a) = \bigwedge \{x \in S : x \geq a\}$.

Let $(\mathrm{Sub}(L), \subseteq)$ be the poset of all sublocales of L ordered by inclusion. In what follows we say that M is a *co-frame* if M^{op} is a frame.

Proposition 2.3 ([7,24,25])**.** *The poset* $(\mathrm{Sub}(L), \subseteq)$ *is a co-frame in which, given* $\mathcal{S} \subseteq \mathrm{Sub}(L)$*, one has* $\bigwedge \mathcal{S} = \bigcap \mathcal{S}$ *and* $\bigvee \mathcal{S} = \{\bigwedge A : A \subseteq \bigcup \mathcal{S}\}$*; and* $\{1\}$ *is the bottom, while* L *is the top of* $(\mathrm{Sub}(L), \subseteq)$*.*

III. The frame of sublocales. For various reasons we will work with the dual lattice of $(\mathrm{Sub}(L), \subseteq)$. We shall denote it by

$$\mathcal{S}(L) = (\mathcal{S}(L), \sqsubseteq),$$

i.e., given $S, T \in \mathrm{Sub}(L)$, we write

$$S \sqsubseteq T \text{ in } \mathcal{S}(L) \quad \text{iff} \quad T \subseteq S \text{ in } \mathrm{Sub}(L),$$

and – consequently – with $\mathcal{T} \subseteq \mathrm{Sub}(L)$ we shall write:

$$\bigsqcup \mathcal{T} = \bigwedge \mathcal{T} \quad \text{and} \quad \bigsqcap \mathcal{T} = \bigvee \mathcal{T}.$$

In $\mathcal{S}(L)$ we let $\perp = L$ (the bottom) and $\top = \{1\}$ (the top). The pseudocomplement of S in the frame $\mathcal{S}(L)$ will standardly be denoted S^*. Thus $S^* = \bigsqcup \{T \in \mathcal{S}(L) : S \sqcap T = \perp\}$.

We note that $a \mapsto \mathfrak{c}(a)$ is a frame embedding $\mathfrak{c} : L \hookrightarrow \mathcal{S}(L)$. The subframe of $\mathcal{S}(L)$ consisting of all closed sublocales will be denoted by $\mathfrak{c}(L)$. Clearly, L and $\mathfrak{c}(L)$ are isomorphic. The isomorphism will also be denoted by $\mathfrak{c}$, so that $\mathfrak{c}^{-1}$ sends $\mathfrak{c}(a)$ back to a.

Recall also that $\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ are complements to each other in $\mathcal{S}(L)$.

IV. The frame of reals. Frames can be specified by giving generators and relations (cf. [15, II.2.11]). There are various definitions of the frame of reals (see e.g. [15,3,4,22]). In [22], the frame $\mathfrak{L}(\mathbb{R})$ of reals has generators of the form $(r,-)$ and $(-,r)$, where $r \in \mathbb{Q}$, subject to the following relations:

(r1) $(r,-) \wedge (-,s) = 0$ whenever $r \geq s$,
(r2) $(r,-) \vee (-,s) = 1$ whenever $r < s$,
(r3) $(r,-) = \bigvee_{s>r}(s,-)$,
(r4) $(-,r) = \bigvee_{s<r}(-,s)$.
(r5) $\bigvee_{r\in\mathbb{Q}}(r,-) = 1$,
(r6) $\bigvee_{r\in\mathbb{Q}}(-,r) = 1$.

A morphism having $\mathfrak{L}(\mathbb{R})$ as a domain will be defined on its generators. Such a map uniquely determines a frame morphism if and only if it makes relations (r1)–(r6) into identities on the codomain frame, say M, which later on will be chosen as $M = \mathcal{S}(L)$.

We shall often use the self-isomorphism $-(\cdot)$ of the hom-set $\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), M)$ defined by

$$(-f)(r,-) = f(-,-r) \quad \text{and} \quad (-f)(-,r) = f(-r,-).$$

V. Localic real functions. Before [12], the lattice-ordered ring $\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)$ (cf. [3]) has not been a part of the lattice of all lower (or upper) semicontinuous real functions on L (since the latter continuities involve domains being certain subframes of $\mathfrak{L}(\mathbb{R})$; cf. [9–11,14]). In order to overcome this inconvenient situation and to have the concept of *an arbitrary not necessarily continuous* real function on L, rather than dealing with the hom-set $\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)$ we will be dealing with

$$\mathrm{F}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(L))$$

(see [12] where the advantage of using $\mathrm{F}(L)$ has been documented). The set $\mathrm{F}(L)$ is partially ordered by the pointwise ordering:

$$f \leq g \quad \text{iff} \quad f(r,-) \sqsubseteq g(r,-) \quad \text{iff} \quad g(-,r) \sqsubseteq f(-,r)$$

for all $r \in \mathbb{Q}$, under which it becomes a lattice (actually, it is a lattice-ordered ring, a fact which is not needed for our purposes; cf. [3] or [12]). Specifically,

$$\begin{aligned}(f \vee g)(r,-) &= f(r,-) \sqcup g(r,-),\\ (f \vee g)(-,r) &= f(-,r) \sqcap g(-,r),\\ f \wedge g &= -((-f) \vee (-g)).\end{aligned}$$

In the frame $\mathcal{S}(L)$ there is enough room to distinguish all types of continuities (cf. [12]). An $f \in \mathrm{F}(L)$ is just thought of as being an *arbitrary real function* on L. The situation is reminiscent of dealing with a real function $X \to \mathbb{R}$ as with a continuous function $\mathcal{D}(X) \to \mathbb{R}$ where $\mathcal{D}(X)$ is the discrete space on the set X.

Definition 2.4 (*[12]*)**.** An $f \in \mathrm{F}(L)$ is *continuous* if $f(r,-)$ is a closed sublocale for all $r \in \mathbb{Q}$ (*lower semicontinuity*) and $f(-,r)$ is a closed sublocale for all $r \in \mathbb{Q}$ (*upper semicontinuity*).

The collection of all continuous members of $\mathrm{F}(L)$ will be denoted by $\mathrm{C}(L)$. Note again that it is a sublattice of $\mathrm{F}(L)$ under the ordering inherited from $\mathrm{F}(L)$.

VI. Generating localic real functions by scales. Let M be an arbitrary frame (in Sections 4 and 5 we shall put $M = \mathcal{S}(L)$). A family $\mathcal{C} = \{c_r : r \in \mathbb{Q}\} \subseteq M$ is said to be an *extended scale* if $c_r \vee c_s^* = 1$ whenever $r < s$. An extended scale becomes a *scale* if $\bigvee \mathcal{C} = 1 = \bigvee \mathcal{C}^*$. Thus, an extended scale is necessarily antitone. We shall also need the following simple observation.

Remark 2.5. Let $\mathcal{C} = \{c_r : r \in \mathbb{Q}\}$ be an antitone subfamily of a frame M. Suppose that for all $r < s$ there is a complemented element $a \in M$ such that $c_s \leq a \leq c_r$. Then $\mathcal{C}$ is an extended scale. [Indeed, one then has $c_r \vee c_s^* \geq a \vee a^* = 1$.]

Lemma 2.6 (*See [12] and cf. [3]*)**.** *Let $\mathcal{C} = \{c_r : r \in \mathbb{Q}\} \subseteq M$ be a scale. Then*

$$f(r,-) = \bigvee_{s>r} c_s \quad \textit{and} \quad f(-,r) = \bigvee_{s<r} c_s^*$$

determine a frame morphism $f : \mathfrak{L}(\mathbb{R}) \to M$.

Lemma 2.7. *Let $f, g : \mathfrak{L}(\mathbb{R}) \to M$ be generated by the extended scales $\mathcal{C} = \{c_r : r \in \mathbb{Q}\}$ and $\mathcal{D} = \{d_r : r \in \mathbb{Q}\}$, respectively. The following hold:*

(0) *$f(r,-) \leq f(-,s)$ and $f(-,s)^* \leq f(r,-)$ whenever $r < s$ in $\mathbb{Q}$.*
(1) *$f(-,r) \leq c_r \leq f(-,r)^*$ for all $r \in \mathbb{Q}$.*
(2) *$f \leq g$ if and only if $c_s \leq d_r$ whenever $r < s$ in $\mathbb{Q}$.*
(3) *Both $\{f(r,-) : r \in \mathbb{Q}\}$ and $\{f(-,r)^* : r \in \mathbb{Q}\}$ are scales that generate f.*

Proof. (0) Let $r < s$. Then $f(r,-)^* = f(r,-)^* \wedge (f(r,-) \vee f(-,s)) = f(r,-)^* \wedge f(-,s)$. Hence the first inequality follows. The second one is dual to the first: $f(-,s)^* = (-f)(-s,-)^* \leq (-f)(-,-r) = f(r,-)$. As for (1) and (2) we refer to [12, Lemma 4.4]. To prove (3), let $\mathcal{C} = \{c_r = f(r,-) : r \in \mathbb{Q}\}$ and $\mathcal{D} = \{d_r = f(-,r)^* : r \in \mathbb{Q}\}$. We have:

$$c_s \leq d_s \leq c_r \leq d_r \quad \text{if } r < s$$

(the second inequality follows from (0)). Thus $\bigvee \mathcal{D} \geq \bigvee \mathcal{C} = 1$ and $\bigvee \mathcal{C}^* \geq \bigvee \mathcal{D}^* \geq \bigvee_{r\in\mathbb{Q}} f(-,r) = 1$. If $r < s$, then

$$c_r \vee c_s^* \geq f(r,-) \vee f(-,s) = 1.$$

Similarly, we have

$$d_r \vee d_s^* = f(-,r)^* \vee f(-,s)^{**} \geq f(r,-) \vee f(-,s) = 1.$$

So, both $\mathcal{C}$ and $\mathcal{D}$ are scales. Let $r < t < s$. Then $c_s \leq d_t \leq c_r$, so that by (2) the two scales generate the same function g, say, and $g(r,-) = \bigvee_{s>r} c_s = \bigvee_{s>r} f(s,-) = f(r,-)$, i.e. $g = f$. □

3. Complete separation of sublocales

According to Ball and Walters-Wayland [1, Def. 6.2.1], given a frame L and quotient maps $h : L \to H$ and $k : L \to K$, the quotients H and K are said to be *completely separated* if there is a frame morphism $\varphi : \mathfrak{L}(\mathbb{R}) \to L$ such that

$$h \circ \varphi(0,-) = 0_H \quad \text{and} \quad k \circ \varphi(-,1) = 0_K.$$

Because, in this paper, we work with sublocales (rather than with quotient frames), we introduce the following:

Definition 3.1. Two sublocales S and T of a frame L are *completely separated* in $\mathcal{S}(L)$ if there is an $f \in \mathrm{C}(L)$ such that

$$f(0,-) \sqsubseteq S \quad \text{and} \quad f(-,1) \sqsubseteq T.$$

Needless to say that one can equivalently require the existence of a $g \in \mathrm{C}(L)$ such that

$$g(r,-) \sqsubseteq S \quad \text{and} \quad g(-,s) \sqsubseteq T$$

for some $r < s$ in $\mathbb{Q}$. Also note that the relation of being completely separated is *symmetric*: S and T are completely separated, then so are T and S.

It is now appropriate to show that the above mentioned concept of [1] is equivalent to that of Definition 3.1. Indeed:

Proposition 3.2. *Let $h : L \to H$ and $k : L \to K$ be quotient maps and let S_H and S_K be the corresponding sublocales. The following are equivalent:*

(1) *There exists a frame morphism $\varphi : \mathfrak{L}(\mathbb{R}) \to L$ such that $h \circ \varphi(0,-) = 0_H$ and $k \circ \varphi(-,1) = 0_K$.*
(2) *There exists a continuous frame morphism $f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ such that $f(0,-) \sqsubseteq S_H$ and $f(-,1) \sqsubseteq S_K$.*

Proof. (1) ⇒ (2): Since $h \circ \varphi(0,-) = 0_H$, we have $S_H = h_*(H) \subseteq \mathfrak{c}(\varphi(0,-))$ (cf. Remark 2.1). Thus, we obtain $f(0,-) \sqsubseteq S_H$ with $f = \mathfrak{c} \circ \varphi$. A similar argument applies to $f(-,1) \sqsubseteq S_K$.

(2) ⇒ (1): If $\mathfrak{c}(a) = f(0,-) \sqsubseteq S_H = h_*(L)$, then by Remark 2.1 we get $h(a) = 0_H$. Further, we have $a = \mathfrak{c}^{-1} \circ f(0,-)$ and $h(a) = h \circ \mathfrak{c}^{-1} \circ f(0,-) = h \circ \varphi(0,-) = 0_H$ where $\varphi = \mathfrak{c}^{-1} \circ f : \mathfrak{L}(\mathbb{R}) \to L$ is a frame morphism. Similarly we get $k \circ \varphi(-,1) = 0_K$. □

To handle complete separation in $\mathcal{S}(L)$ we introduce an extra order in $\mathcal{S}(L)$ as follows:

Notation. Let L be a frame. Given $S, T \in \mathcal{S}(L)$, we write

$$S \Subset T$$

iff there exists a continuous $f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ such that

$$S \sqsubseteq f(-,1)^* \quad \text{and} \quad f(0,-) \sqsubseteq T. \qquad (\Subset)$$

We may also write $S \Subset_f T$ to indicate the function f in ($\Subset$). It is clear that it is enough to find a $g \in \mathrm{C}(L)$ such that $S \sqsubseteq g(-,s)^* \sqsubseteq g(r,-) \sqsubseteq T$ for some $r < s$ in $\mathbb{Q}$. This, in turn, is equivalent to requiring the existence of an $h \in \mathrm{C}(L)$ such that $S \sqsubseteq h(r,-)^* \sqsubseteq h(-,s) \sqsubseteq T$ for some $r < s$ in $\mathbb{Q}$. The reader should keep in mind that both $f(-,s)$ and $f(r,-)$ are closed sublocales, hence complemented in $\mathcal{S}(L)$.

Remark 3.3. Let $S, T \in \mathcal{S}(L)$. Then $S \Subset T$ if and only if T and S^* are completely separated in $\mathcal{S}(L)$. [We only need to notice that, given an $f \in \mathrm{C}(L)$ one has: $S \sqsubseteq f(-,1)^*$ iff $f(-,1) \sqsubseteq S^*$.]

Properties 3.4. *For $S, S_i, T, T_i \in \mathcal{S}(L)$ ($i = 1, 2$), the following hold:*

(1) *$S \Subset T$ implies $S \sqsubseteq T$,*
(2) *$S \sqsubseteq S_1 \Subset T_1 \sqsubseteq T$ implies $S \Subset T$,*
(3) *$S_1 \Subset T$ and $S_2 \Subset T$ imply $S_1 \sqcup S_2 \Subset T$,*
(4) *$S \Subset T_1$ and $S \Subset T_2$ imply $S \Subset T_1 \sqcap T_2$,*
(5) *$S \Subset T$ implies $S \Subset \mathfrak{c}(a) \Subset T$ for some $a \in L$.*

Proof. (1) and (2) are obvious. For (3), if $S_i \Subset_{f_i} T$ $(i = 1, 2)$, then $S_1 \sqcup S_2 \Subset_{f_1 \vee f_2} T$, and similarly for (4). To show (5), let $S \Subset T$. If $s < t < r$, then $\mathfrak{c}(a) = f(t, -)$ is as required, for $S \sqsubseteq f(-, s)^* \sqsubseteq f(t, -) \sqsubseteq f(-, t)^* \sqsubseteq f(r, -) \sqsubseteq T$. □

Given a frame M and $a, b \in M$, a is *really inside* b (written: $a \prec\!\prec b$) if there exists a family $\{c_r : r \in \mathbb{Q} \cap [0, 1]\} \subseteq M$ such that $a \leq c_0, c_1 \leq b$ and $c_r^* \vee c_s = 1$ when $r < s$.

Remark 3.5. Let $a, b \in L$. Then $a \prec\!\prec b$ if and only if $\mathfrak{c}(a) \Subset \mathfrak{c}(b)$.

Proof. By [15, Proposition IV-1.4], $a \prec\!\prec b$ iff there is a frame morphism $\varphi : \mathfrak{L}(\mathbb{R}) \to L$ such that $a \leq \varphi(0, -)^*$ and $\varphi(-, 1) \leq b$. Hence $a \leq \varphi(0, -)^* \leq \varphi(-, \frac{1}{2})$ and $\varphi(-, 1) \leq b$. It follows that there is an $f = \mathfrak{c} \circ \varphi \in \mathrm{C}(L)$ such that $\mathfrak{c}(a) \sqsubseteq f(-, \frac{1}{2}) \sqsubseteq f(\frac{1}{2}, -)^* \sqsubseteq f(-, 1) \sqsubseteq \mathfrak{c}(b)$, i.e. $\mathfrak{c}(a) \Subset \mathfrak{c}(b)$.

Conversely, if $\mathfrak{c}(a) \Subset_f \mathfrak{c}(b)$, then $\psi = \mathfrak{c}^{-1} \circ f : \mathfrak{L}(\mathbb{R}) \to L$ is a frame morphism satisfying $a \leq \psi(-, s)^* \leq \psi(r, -) \leq b$ for some $r < s$. □

Remarks 3.6. (1) A frame L is called *completely regular* if $a = \bigvee\{b \in L : b \prec\!\prec a\}$ for all $a \in L$. The latter is thus equivalent to the requirement that $S = \bigsqcup\{T \in \mathfrak{c}(L) : T \Subset S\}$ for each $S \in \mathfrak{c}(L)$.

(2) It may be remarked that if $a \leq b$ in L and a or b is complemented, then $a \prec\!\prec b$. In particular, in $M = \mathcal{S}(L)$ we clearly have:

$$\mathfrak{c}(a) \prec\!\prec \mathfrak{c}(b) \quad \text{iff} \quad \mathfrak{c}(a) \sqsubseteq \mathfrak{c}(b) \quad \text{iff} \quad a \leq b.$$

Thus if $a \leq b$ and a fails to be really inside b, then $\mathfrak{c}(a) \not\Subset \mathfrak{c}(b)$. This shows that on $\mathcal{S}(L)$, the relation $\Subset$ is generally stronger than $\prec\!\prec$.

4. Localic insertion theorem

Katětov [17] has a relation ρ on a power set which makes sense on an arbitrary complete lattice and which, denoted by $\Subset$, can equivalently be described as follows (cf. [13,18,19], see also [2]).

Let M be an arbitrary complete lattice. A binary relation $\Subset$ on M is called a *Katětov relation* if it satisfies the following conditions for all $a, b, c, d \in M$:

($\Subset_1$) $a \Subset b$ implies $a \leq b$,
($\Subset_2$) $c \leq a \Subset b \leq d$ implies $c \Subset d$,
($\Subset_3$) $a, b \Subset c$ implies $a \vee b \Subset c$,
($\Subset_4$) $a \Subset b, c$ implies $c \Subset b \wedge c$,
($\Subset_5$) [Insertion property] $a \Subset b$ implies $a \Subset c \Subset b$ for some $c \in L$.

The next lemma is of basic importance to what follows.

Lemma 4.1 ([13,19]). *Let M be a complete lattice endowed with a Katětov relation $\Subset$. Let $\{a_r : r \in \mathbb{Q}\}$ and $\{b_r : r \in \mathbb{Q}\}$ be antitone families of M such that $a_s \Subset b_r$ if $r < s$. Then there exists a family $\{c_r : r \in \mathbb{Q}\}$ such that $a_s \Subset c_q \Subset c_p \Subset b_r$ whenever $r < p < q < s$.*

We are now in a position to state the localic analogue of the insertion theorem of Blair [5] and Lane [21] (also cf. [20]).

Theorem 4.2 (*Localic Insertion Theorem*). *Let L be a frame and let $g, h \in \mathrm{F}(L)$. The following statements are equivalent:*

(1) *There exists $f \in \mathrm{C}(L)$ such that $g \leq f \leq h$.*
(2) *The sublocales $g(-, s)$ and $h(r, -)$ are completely separated in $\mathcal{S}(L)$ for every $r < s$ in $\mathbb{Q}$.*

Proof. (1) ⇒ (2): Given $r < t < s$ in $\mathbb{Q}$, we have

$$g(t, -) \sqsubseteq f(t, -) \sqsubseteq f(-, t)^* \sqsubseteq f(r, -) \sqsubseteq h(r, -),$$

i.e. $g(t, -) \Subset_f h(r, -)$. Thus, there exists $k \in \mathrm{C}(L)$ such that $g(t, -) \sqsubseteq k(-, 1)^* \sqsubseteq k(0, -) \sqsubseteq h(r, -)$. Further, $k(-, 1) \sqsubseteq k(-, 1)^{**} \sqsubseteq g(t, -)^* \sqsubseteq g(-, s)$. We have shown that

$$k(0, -) \sqsubseteq h(r, -) \quad \text{and} \quad k(-, 1) \sqsubseteq g(-, s),$$

i.e. and $g(-, s)$ and $h(r, -)$ are completely separated in $\mathcal{S}(L)$.

(2) ⇒ (1): Let $r < s$. We first notice that since, $g(-, s) \sqsubseteq g(s, -)^*$ and $h(r, -) \sqsubseteq h(-, r)^*$, the sublocales $g(s, -)^*$ and $h(-, r)^*$ are thus completely separated too. By Remark 3.3 we have:

$$g(s, -) \Subset h(-, r)^* \quad \text{if } r < s.$$

We can now employ Lemma 4.1 (with $(M, \Subset) = (\mathcal{S}(L), \Subset)$, $a_s = g(s, -)$, and $b_r = h(-, r)^*$). Let $\mathcal{C} = \{C_r : r \in \mathbb{Q}\} \subseteq \mathcal{S}(L)$ be the family satisfying the assertion of Lemma 4.1, i.e.

$$g(s, -) \Subset C_q \Subset C_p \Subset h(-, r)^* \tag{I}$$

for all $r < p < q < s$. By Lemma 2.7(3) we get

$$\bigsqcup \mathcal{C} = \top = \bigsqcup \mathcal{C}^*.$$

Also, $C_q \Subset C_p$ together with Remark 2.5 and Property 3.4(5) yields that $\mathcal{C}$ is a scale. The function f generated by it satisfies $g \leq f \leq h$ on account of (I) and Lemma 2.7(2). □

Remark 4.3. Sample examples of corollaries of Theorem 4.2 are in [12, Theorems 8.1 and 8.7]. One may easily have further special cases by introducing various kinds of frames that have not yet been introduced in the literature. The interested reader may consult a long list of corollaries in the spatial case given by Lane [21].

5. Localic extension theorem

Our next goal will be to develop a localic analogue of the Topological Extension Theorem of Mrówka [23] (see the Introduction). Unfortunately, we have not been able to prove it for arbitrary sublocales. We need the following lemma:

Lemma 5.1. (1) *A sublocale T of a sublocale S of a frame L is a sublocale of L.*
(2) *A closed (resp. open) sublocale T of a complemented sublocale S of a frame L is a complemented sublocale of L.*

Proof. For (1) see, e.g., [24]. We now check (2) for the case of a closed T. Then $T = \mathfrak{c}(a) \sqcup S$ for some a in L. Then

$$T \sqcup T^* = \mathfrak{c}(a) \sqcup S \sqcup (\mathfrak{o}(a) \sqcap S^*) = (\mathfrak{c}(a) \sqcup S \sqcup \mathfrak{o}(a)) \sqcap (\mathfrak{c}(a) \sqcup S \sqcup S^*) = \top.$$

If T is open, then $T = \mathfrak{o}(a) \vee S$ for some a in L and the above calculation remains the same. □

Let S be a sublocale of L. Following [12], we say that $\tilde{f} \in \mathrm{C}(L)$ is a *continuous extension* of $f \in \mathrm{C}(S)$ iff for all $r \in \mathbb{Q}$ one has:

$$f(r, -) = S \sqcup \tilde{f}(r, -) \quad \text{and} \quad f(-, r) = S \sqcup \tilde{f}(-, r).$$

We let $\mathrm{C}^*(L) = \{f \in \mathrm{C}(L) : f(-, 0) \sqcup f(1, -) = \perp\}$, the collection of all *bounded* continuous localic real functions.

Theorem 5.2 (*Localic Extension Theorem*)**.** *Let L be a frame, S be a complemented sublocale of L and let $f \in \mathrm{C}^*(S)$. The following are equivalent:*

(1) *There exists a continuous extension of f to the whole of L.*
(2) *The sublocales $f(-, r)$ and $f(s, -)$ are completely separated in $\mathcal{S}(L)$ for all $r < s$ in $\mathbb{Q}$.*

Proof. (1) ⇒ (2): Let $\bar{f} \in \mathrm{C}^*(L)$ be the extension of $f \in \mathrm{C}^*(S)$. By Lemma 5.1(1), $f(s, -)$ and $f(-, r)$ are sublocales of L for each $r < s$ in $\mathbb{Q}$ and so we have that:

$$\bar{f}(s, -) \sqsubseteq f(s, -) \quad \text{and} \quad \bar{f}(-, r) \sqsubseteq f(-, r).$$

(2) ⇒ (1): Let $f \in \mathrm{C}^*(S)$ with S a complemented sublocale of L. Define

$$S_r = \begin{cases} \top & \text{if } r < 0, \\ f(r, -) & \text{if } 0 \le r < 1, \\ \perp & \text{if } r \ge 1, \end{cases} \quad \text{and} \quad T_r = \begin{cases} \top & \text{if } r < 0, \\ f(-, r)^* & \text{if } 0 \le r < 1, \\ \perp & \text{if } r \ge 1. \end{cases}$$

Since $f(r, -)$ (resp. $f(-, r)^*$) is a closed (resp. open) sublocale of the complemented sublocale S for each $r \in \mathbb{Q}$, it follows from Lemma 5.1(2) and Remark 2.5 that $\{S_r\}_{r\in\mathbb{Q}}$ and $\{T_r\}_{r\in\mathbb{Q}}$ are scales in $\mathcal{S}(L)$ which generate f_1 and f_2 in $\mathrm{C}^*(L)$, respectively. Let $r < s$. If $r < 0$, then $T_s \sqsubseteq \top = S_r$, if $s \ge 1$, then $T_s = \perp \sqsubseteq S_r$ and finally, if $0 \le r < s < 1$, then $T_s = f(-, s)^* \sqsubseteq f(r, -) = S_r$. Hence $f_2 \le f_1$. If $r < s$, then $f_2(s, -)$ and $f_1(-, r)$ are completely separated, so by Theorem 4.2 there is an $h \in \mathrm{C}^*(L)$ such that $f_2 \le h \le f_1$. Finally, we conclude that h is the desired extension of f. Indeed, for $r < 0$ we have $f(r, -) = \top_S = \top = S \sqcup \top = S \sqcup h(r, -)$ and for $r \ge 1$ we have $f(r, -) = \perp_S = S = S \sqcup \perp = S \sqcup h(r, -)$. Finally for each $0 \le r < 1$ we have

$$\begin{aligned} f(r, -) &= \bigsqcup_{1>s>r} f(s, -) \sqsubseteq \bigsqcup_{1>s>r} f(-, s)^* = \bigsqcup_{1>s>r} T_s = f_2(r, -) \\ &\sqsubseteq h(r, -) \sqsubseteq S \sqcup h(r, -) \sqsubseteq S \sqcup f_1(r, -) = S \sqcup f(r, -) = f(r, -) \end{aligned}$$

where the latter equality holds on account of $f(r, -)$ being a sublocale of S. Similarly, for $r \le 0$ we have $f(-, r) = \perp_S = S = S \sqcup \perp = S \sqcup h(-, r)$, for $r > 1$ we have $f(r, -) = \top_S = \top = S \sqcup \top = S \sqcup h(-, r)$ and for each $0 < r \le 1$ we have

$$\begin{aligned} f(-, r) &= \bigsqcup_{0<t<r} f(-, t) \sqsubseteq \bigsqcup_{0<t<r} f(t, -)^* = \bigsqcup_{0<t<r} S_t^* = f_1(-, r) \\ &\sqsubseteq h(-, r) \sqsubseteq S \sqcup h(-, r) \sqsubseteq S \sqcup f_2(-, r) = S \sqcup f(-, r) = f(-, r). \quad \square \end{aligned}$$

Among special cases of Theorem 5.2 is Tietze extension theorem for frames (as formulated in [12] and not as in [22]).

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